Solutions for Go Figure 1999

1. A sequence where every term after the first term is a fixed number times the term just before it, is called a *geometric progression* and the fixed multiplier is called the *ratio*. Given a term in the progression, we can compute the term immediately *preceeding* that term by dividing by the ratio. Thus $b = 6/3 = 2$ and $a = 2/3$.

2. For part (a) we recognize that the ith term (for $i = 1, 2, \ldots$) is equal to $7i$. In this problem, each of the three progressions has the same difference between terms: 7. Thus the difference between the 100th term of the first progression and the 100th term of the second progression is 2, the same as the difference between the first two terms: $7 - 5$.

   (a) $7 \times 100 = 700$,
   (b) $700 - 2 = 698$.
   (c) $18 - 7 = 11$, so the 100th term is $700 + 11 = 711$.

3. $48 = 4 \times 12$, and $4 = 2 \times 2$. Thus the positive integral factors of 48 are the positive integral factors of 12, as listed in the problem statement, as well as each of these factors multiplied by 2 or 4: $1, 2, 3, 4, 6, 8, 12, 16, 24, 48$.

4. This set is small enough that we can check all cases. However, one observation can eliminate a lot of cases: the largest product cannot have 1 as one of the factors. Multiplying by 1 cannot increase the product. It is always better to use the 1 to increase the size of one of the other factors. Using this observation, there are only 6 remaining possibilities in the list: 6×2, 5×3, 4×4, $4\times2\times2$, $3\times3\times2$, and $2\times2\times2\times2$. Of these, the largest is $3 \times 3 \times 2 = 18$.

5. There are many ways to solve this problem. You can cross multiply each pair of fractions to see which is bigger. You can find a common denominator. However, in this case, the easiest strategy is to find a common numerator. Choosing a least common numerator of 60, and maintaining the original order, the five fractions, become:

$$\frac{60}{84}, \frac{60}{85}, \frac{60}{81}, \frac{60}{82}, \frac{60}{83}.$$

   If two fractions have a common numerator, the one with the smallest denominator is largest.

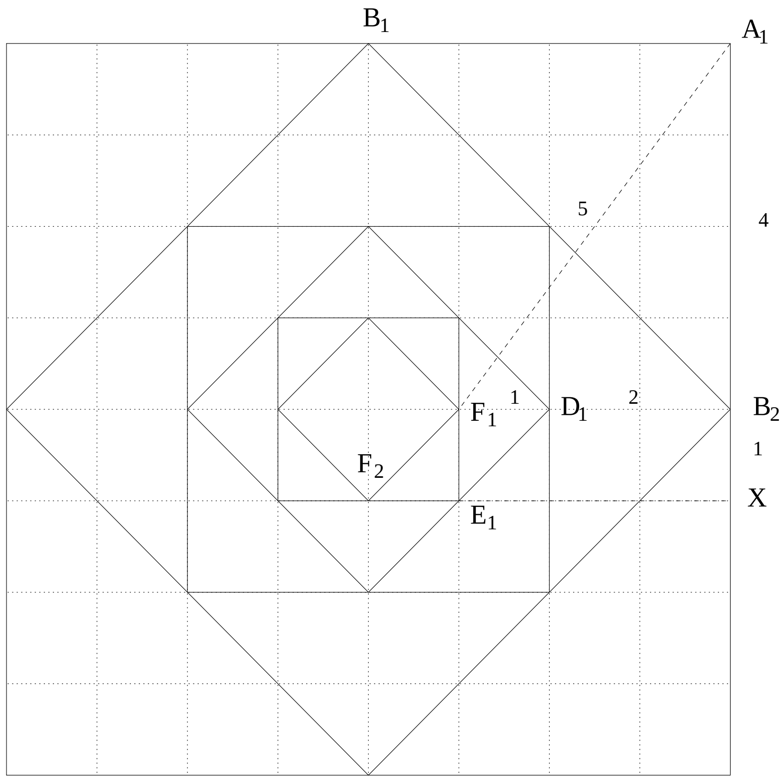
   (a) $\frac{60}{85} = \frac{12}{17}$.
   (b) $\frac{60}{81} = \frac{20}{27}$

6. The unit digit of the product $CCC8$ is the unit digit of $3 \times B$ (product of the unit digits of the two factors). There is only one product of 3 and a single digit that gives a unit digit of 8, namely $3 \times 6 = 18$. Therefore $B = 6$ and the second factor is 26. Because we are multiplying 26 by a 2-digit number, the final product is no more than $26 \times 100 = 2600$. Therefore there are only two possibilities for the final product: 1118 and 2228. Dividing each of these by 26, we find that 26 doesn't evenly divide 2228, but it does evenly divide 1118 and the quotient is of the proper form. Therefore $A = 4$, $B = 6$, $C = 1$.

7. $2 \times 3^9 = 39366$. See solution to problem **??**.

8. In the solution to problem 4, we saw that we do not have to consider any products with 1 as a factor; it is always better to add the 1 to another factor. Similarly, we can eliminate almost all of the possibilities. Instead of 9 we could use $3 \times 3 \times 3 = 27$. This has the same sum, but the product is $27 > 9$. Similarly 8 can be replaced by $2 \times 2 \times 2 \times 2 = 16$, 7 can be replaced by $3 \times 2 \times 2 = 12$, 6 can be replaced by $3 \times 3 = 9$, 5 can be replaced by $3 \times 2 = 6$. We can replace 4 with 2×2. It does not give a strictly bigger number, but it further reduces the set of digits we might use. We now have only 2's and 3's left. We should never have more than two 2's because three 2's sum to 6 and it is better to take $3 \times 3 = 9$ than $2 \times 2 \times 2 = 8$. Since $29 = 3 \times 9 + 2$, the largest possible product of numbers whose sum is 29 is $3^9 \times 2 = 39366$.

9. Consider the problem laid out on a checkerboard which has 8 squares on a side. See the figure below for point labels.



(a) First we calculate the length of a side of square B. Triangle $B_1A_1B_2$ is a right isosceles triangle. Segment B_1A_1 has length 4 (half the length of a side of A), and therefore by the Pythagorean theorem, the length of B_1B_2 is $\sqrt{4^2+4^2} = 4\sqrt{2}$. This is a side of square B and therefore square B has area 32, half the area of square A. Since we do the same operation to create square C from B, the area of C is 16, half the area of B.

(b) From part (a) we know that the length of a side of square C is 4. It has the same center point as square A. Therefore segment D_1B_2 has length 2 (half the difference between the length of the side of square A and length of the side of square C). By a similar argument, or realizing that all corresponding lengths are halved each time we iterate this construction, we see that the length of segment $F_1D_1 = 1$. Therefore the length of $F_1B_2 = 3$. By the Pythagorean theorem, the length of $A_1F_1 = \sqrt{3^2 + 4^2} = 5$.

(c) We must find the length of segments XF_2 and A_1X to apply the Pythagorean theorem. Segment XE_1 is the same length as F_1B_2 (since $E_1F_1B_2X$ is a rectangle). Segment E_1F_2 is half the length of a side of square E. Square E has side length 2 (one quarter the side of A), so E_1F_2 is 1 and the length of $XF_2 = 3 + 1 = 4$. The length of B_2X is the same as the length of E_1F_1 (opposite sides of a rectangle) which in turn is equal to the length of F_2E_1 by construction. Thus B_2X is of length 1 and A_1X is of length $4 + 1 = 5$. Thus the length of AF_2 is $\sqrt{5^2 + 4^2} = \sqrt{41}$.

10. (a) These are all the natural numbers less than $8 = 2^3$. We can therefore use the place values for a binary representation of a number. The desired multiset is $\{1, 2, 4\}$.

(b) Using the same argument as above, we use the place values for the binary representations of the numbers less than $32 = 2^6$. The desired multiset is $\{1, 2, 4, 8, 16\}$.

(c) The example in the problem statement shows that the multiset $\{1, 1, 1, 5\}$ generates 7 numbers. Now consider adding the number 7. We can have 7 by itself (which is not a new number), or add 7 to each of these previously-generated numbers. We get another duplicate (8) and 6 new ones for a total if 13.

(d) Notice that the numbers added to the multiset in problem (c) are all powers of 5, namely $25 = 5^2$, $125 = 5^3$, $625 = 5^4$ and $3125 = 5^5$. The largest number generated by the other numbers in the set is 15 (from problem (b) above). Therefore adding 25 to any of these numbers will generate a new number, and 25 itself will generate a new number. This is also true if we add 50. If we think of the base-5 representation of a number, this representation can use zero, one, two, three, or four 25's. This multiset allows us to use zero, one, or two. Therefore the multiset $\{1, 1, 1, 5, 7, 25, 25\}$ generates $3 \times 13 + 2 = 41$ numbers. The largest number generated is less than 125, so again, in analogy to the base-5 representation, we could have zero or one 125 added to the previous multiset. Thus multiset $\{1, 1, 1, 5, 7, 25, 25, 125\}$ generates $2 \times 41 + 1 = 83$ numbers. Similarly $\{1, 1, 1, 5, 7, 25, 25, 125, 625\}$ generates $2 \times 83 + 1 = 167$ numbers and finally $\{1, 1, 1, 5, 7, 25, 25, 125, 625, 3125\}$ generates $2 \times 167 + 1 = 335$ numbers.